On the Volume Conjecture for hyperbolic Dehn-filled 3-manifolds along the twist knots

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- Calculation of RT invariants
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Main theorems

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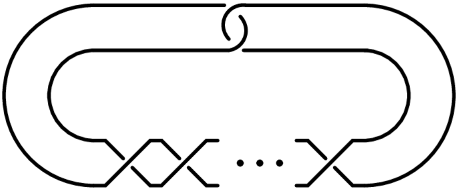
Main theorems

Theorem (Ge-Meng-W.-Yang, 2024)

Let $M = \mathcal{K}_s(p,q)$ be the manifold obtained by S^3 doing p/q Dehn-Surgery along the twist knot $\mathcal{K}_s$. Then there exists a N ($N = 10$ is enough.), so that if $|p|, |q|, |s| > N$,

$$\lim_{r\to\infty, r \text{ odd}} \frac{4\pi}{r} \log \mathrm{RT}_r(\mathcal{K}_s(p,q))$$
$$= \mathrm{Vol}(\mathcal{K}_s(p,q)) + \sqrt{-1}\,\mathrm{CS}(\mathcal{K}_s(p,q)) \ (mod \sqrt{-1}\pi^2\mathbb{Z}),$$



Main theorems

Due to the realtionship between Reshetikhin-Turaev invariants and Turaev-Viro invariants,

Theorem

For a closed,oriented 3-manifolds M

$$\mathrm{TV}_r(M) = 2^{b_2(M)-b_0(M)+2}|\,\mathrm{RT}_r(M)|^2 \quad \forall r$$

where $b_i(M)$ is the i-th $\mathbb{Z}_2$ betti number of M.

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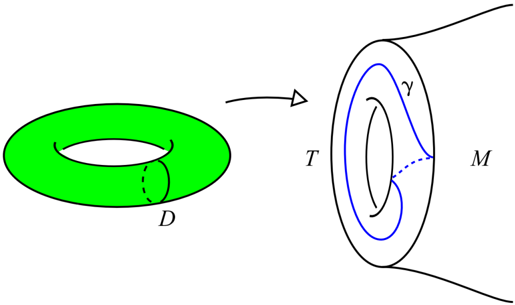
we can directly obtain

$$\lim_{r\to\infty, r \text{ odd}} \frac{2\pi}{r} \log \mathrm{TV}_r(\mathcal{K}_s(p,q)) = \mathrm{Vol}(\mathcal{K}_s(p,q))$$

Dehn Surgery

Definition (Dehn Filling)

Along the boundary torus T of $S^3 \backslash K$, we glue in one solid torus H by a homeomorphism that kills curve $pm_T + ql_T$ on T and meridian m on ∂H, This process is called Dehn filling. m_T, l_T are the meridian and longitude that generate $H_1(\partial(S^3 \backslash K), \mathbb{Z})$. And $p/q \in \mathbb{Q} \cup \{\infty\}$ is called Dehn coefficient.



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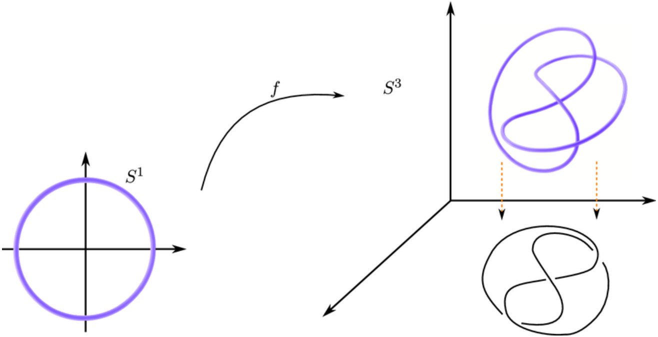
Knots in S^3

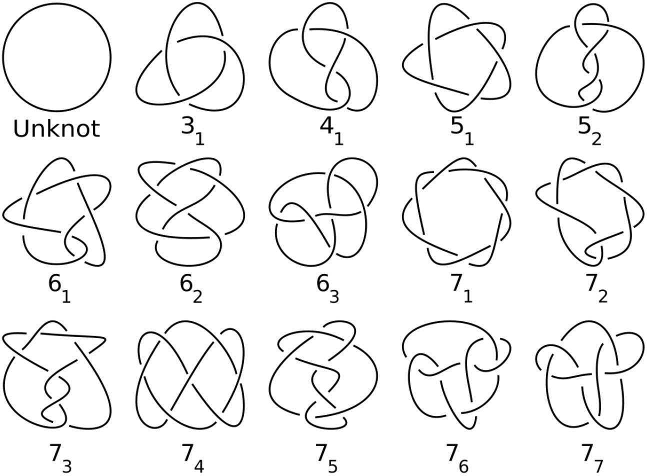
Definition (Knot in S^3)

A knot is a smooth embedding of S^1 into S^3.



Knot diagrams

Example



Knot invariants

Algebraic Invariants

Crossing Number
Bridge Number
Seifert Matrix
Signature
Tricolorability
...

Geometric Invariants

Hyperbolic Volume
Gromov Norm
Chern-Simons Invariants
Slice Genus
...

Analytic Invariants

L^2-Betti Number
L^2-Torsion
L^2 -Spectral Invariants
...

Finite Type Invariants

Vassiliev Invariants
Kontsevich Invariant
...

Volume Conjecture

Quantum Invariants

Polynomial Invariants

(colored) Jones Polynomial
(colored) HOMFLY-PT Polynomial
(colored) Kauffman Polynomial

Reshetikhin-Turaev Invariants
Turaev-Viro Invariants
...

Homological Invariants

Atiyah-Floer Conjecture

Khovanov Homology
Heegaard Floer Homology
Knot Instanton Homology
...

Quantum invariants

Idea of quantum groups is originated from quantum integrable system(QISM). In 1981 **Kulish-Reshetikhin** gave the first example of quantum groups and then in 1986 **Drinfeld-Jimbo** formalized the above idea to the general definition of quantum groups.

Steps for constructing invariants from quantum groups:

- Let $U_q(\mathfrak{g})$ be the quantum universal enveloping algebra of a finite dimensional complex semi-simple Lie algebra $\mathfrak{g}$

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- For each knot component, we associate it an irreducible representation V of $U_q(\mathfrak{g})$
- The quantum invariants can be obtained by taking the quantum trace of endomorphism.

Quantum invariants

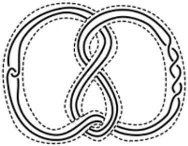
Quantum Invariants	**Quantum Groups**
(colored) Jones polynomial	$U_q(sl_2)$
(colored) HOMFLY-PT polynomial	$U_q(sl_N)$ ($N \geq 2$)
(colored) Kauffman polynomial	$U_q(so_{2N+1})$
Chern-Simons invariants	$U_q(sl_N)$
Reshetikhin-Turae invariants	$U_q(\mathfrak{g})$
Turaev-Viro invariants	$U_q(sl_2)$ or $U_q(so_3)$

The colored version of invariants can be seen as an extension from the **fundamental representation** of quantum groups to **any irreducible representation** of quantum groups.

Knot Classification

Theorem (Thurston)

A knot in S^3 *is hyperbolic if and only if it is not a torus knot or a satellite knot.*



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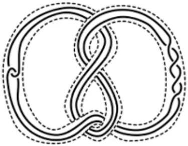


Claim: The vast majority of knots are hyperbolic. Among all prime knots with crossing ≤ 16, there are 13 torus knots, 20 satellite knots and 1701903 hyperbolic knots.

Mostow–Prasad Rigidity

Hyperbolic volume is a invariant of hyperbolic knots by Mostow Rigidity.

Theorem (Mostow Rigidity Theorem)

For $n \geq 3$, any two complete, finite-volume hyperbolic manifolds of dimension n with isomorphic fundamental groups are isometric.

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For non-hyperbolic knots, we use Gromov norm instead of its hyperbolic volume.

Gromov norm

Definition (Jaco-Shalen-Johannson decomposition)

Let K be a knot. Then $S^3 \backslash K$ can be uniquely decomposed into hyperbolic pieces and Seifert fibered pieces by a system of essential tori

$$S^3 \backslash K = \left(\bigcup H_i\right) \cup \left(\bigcup E_j\right)$$

with H_i hyperbolic and E_j Seifert-fibered.

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Definition (Gromov norm)

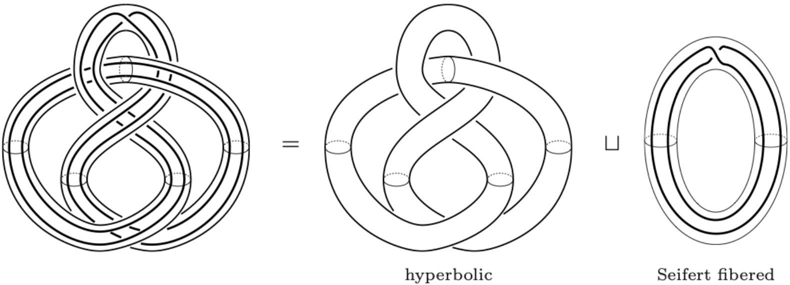
The Gromov norm $Vol(S^3 \backslash K)$ is defined as

$$Vol(S^3 \backslash K) = \sum_i Vol(H_i)$$

JSJ-decomposition

Example

The JSJ decomposition of the (2,1)-cable of the figure-eight knot



Volume

Definition (dilogarithm function)

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt$$

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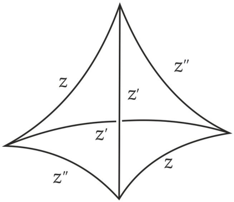
Definition (Bloch-Wigner function)

$$\mathrm{D}_2(z) = \Im(\mathrm{Li}_2(z)) + \log|z| \Im \log(1-z), \text{ if } z \in \mathbb{C} \backslash \{0, 1\}$$

Volume

Proposition

For a ideal tetrahedron T with shape $z \in \mathbb{H}$, $\mathrm{Vol}(T) = \mathrm{D}_2(z)$



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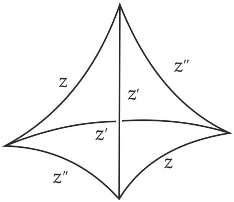

After an geometric ideal triangulation of a cusped hyperbolic 3-manifold M, we have $\mathrm{Vol}(M) = \sum_{T} \mathrm{Vol}(T)$.

Chern-Simons

The **Chern-Simons invariant** is a topological invariant associated with 3-manifolds, defined using a connection A on a principal G-bundle over the manifold.

- Defined for closed, oriented 3-dimensional manifolds.
- Defined via a connection 1-form A on a principal G-bundle, where G is a Lie group (often $G = SU(2)$ or $G = SO(3)$).
- The Chern-Simons invariant depends on the choice of the connection, but moduli space analysis shows that its variation gives a topological invariant for flat connections.

Chern-Simons

Definition (Chern-Simons)

Let M be a closed, oriented 3-manifold, and A be a connection 1-form on a principal G-bundle over M. The Chern-Simons invariant $CS(A)$ is defined as:

$$CS(A) = \frac{1}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \tag{1}$$

where Tr is the trace over the Lie algebra $\mathfrak{g}$ of G.

This integral represents a 3-dimensional characteristic class, derived from the Chern-Weil theory, and captures topological information about the 3-manifold.

Chern-Simons

Under a gauge transformation $g : M \to G$, the connection transforms as $A \mapsto A^g = g^{-1}Ag + g^{-1}dg$. The Chern-Simons invariant transforms as:

$$CS(A^g) = CS(A) + 2\pi n,$$

where $n \in \mathbb{Z}$ is an integer that depends on the winding number of the gauge transformation. This integer ambiguity reflects the fact that the Chern-Simons functional is not gauge-invariant, but its variation is.

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Thus, the Chern-Simons invariant mod 2π defines a well-defined topological invariant.

KMM Volume Conjecture

Volume conjecture: **quantum topology** $\Leftrightarrow$ **hyperbolic geometry**.

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Conjecture (Kashaev 1997; Murakami-Murakami 2001)

For any hyperbolic knot K in S^3

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K; e^{\frac{2\pi\sqrt{-1}}{N}})|}{N} = Vol(S^3 \backslash K)$$

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Conjecture (Complexification)

For any hyperbolic knot K in S^3

$$2\pi \lim_{N\to\infty} \frac{\log J_N(K; e^{\frac{2\pi\sqrt{-1}}{N}})}{N} = Vol_{\mathbb{C}}(S^3 \backslash K)$$

where $Vol_{\mathbb{C}}(S^3 \backslash K) = Vol(S^3 \backslash K) + \sqrt{-1}CS(S^3 \backslash K) \ (mod\ \sqrt{-1}\pi^2\mathbb{Z})$.

Some Progresses of KMM Volume Conjecture

1. Kashaev and Yokota: 5_2
2. Ohtsuki, Yokota and Takata: $6_i, 7_i, 8_6, 8_{12}$
3. Garoufalidis and Le: Borromean rings
4. Kashaev and Tirkkonen: Torus knots
5. Hikami: torus links of type $(2, 2m)$
6. Zheng: Whitehead doubles of torus knots, Twisted Whitehead links
7. van der Veen: knots and links with volume 0, Whitehead chains
8. Yamazaki and Yokota: a satellite link around the figure-eight knot with pattern the Whitehead link
9. Chen and Zhu: twist knots $\mathcal{K}_p$ with $p \geq 6$, 2023, arxiv.

Framed link

Framed link can be seen as a “thickening” of each component of the link into a ribbon around the curve.

Reshetikin-Turaev and Turaev-Viro invariants

Reshetikhin-Turaev invariants are a family of quantum invariants of framed links which can be constructed from representations of ribbon Hopf algebra. Nicolai Reshetikhin and Vladimir Turaev discovered these invariants in 1991 to rigorously realize Witten's proposed invariants from quantum field theory(QFT).

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For manifolds possibly with boundarys: **Turaev-Viro** invariants.

Chen-Yang Volume Conjecture

Conjecture (Chen-Yang: RT, 2018, Quantum Topol.)

For any hyperbolic closed orientable 3-manifold M

$$4\pi \lim_{r\to\infty, r \text{ is odd}} \frac{\log RT_r(M; q(2) = e^{\frac{2\sqrt{-1}\pi}{r}})}{r} = Vol_{\mathbb{C}}(M)$$

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Conjecture (Chen-Yang: TV, 2018, Quantum Topol.)

For any hyperbolic orientable 3-manifold M, either closed, with cusps, or compact with totally geodesic boundary

$$2\pi \lim_{r\to\infty, r \text{ is odd}} \frac{\log TV_r(M; q(2)) = e^{\frac{2\sqrt{-1}\pi}{r}})}{r} = Vol(M)$$

Recent Progress of Chen-Yang Volume Conjecture

Some Manifolds for which the Volume Conjecture is proved:

1. (Algebr. Geom. Topol. 2018)
   Ohtsuki: Reshetikin-Turaev invariants of integral Dehn filled manifolds along figure-eight knot
2. (arXiv 2020)
   Wong-Yang: Reshetikin-Turaev invariants of rational Dehn filled manifolds along figure-eight knot
3. (Quantum Topol. 2018)
   Detcherry, Kalfagianni and Yang: (modified) Turaev-Viro invariant of complements of 4_1 and Borromean ring
4. (J. Differential Geom. 2022)
   Belletti, Detcherry, Kalfagianni and Yang: Turaev-Viro invariant of fundamental shadow link in M_c
5. (arXiv 2024)
   Chen-Zhu: Reshetikin-Turaev invariants of integer Dehn filled manifolds along the twist knots

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Sketch of our Paper

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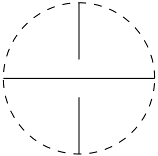
3. We observed a **"big cancellation"** between the Fourier coefficients. And we use the saddle point method to estimate the Fourier coefficients.

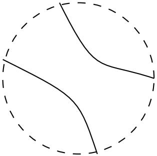
Skein approach of RT

The Kauffman bracket skein module $\mathrm{K}_r(A \times [0,1])$ is a $\mathbb{C}$-module generated by framed link diagrams in $A \times [0,1]$ modulo the following two relations:

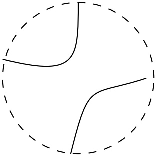
1. Kauffman Bracket Skein Relation:



$$= e^{-\frac{\pi\sqrt{-1}}{r}}$$



$$+ \quad e^{\frac{\pi\sqrt{-1}}{r}}$$



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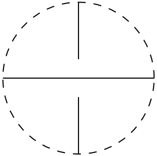
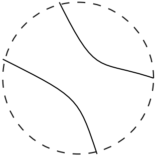


$= e^{-\frac{\pi\sqrt{-1}}{r}}$



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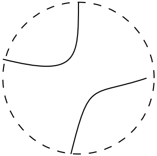

2. Framing Relation:

$$L \cup \bigcirc = \left(-e^{\frac{2\pi\sqrt{-1}}{r}} - e^{-\frac{2\pi\sqrt{-1}}{r}}\right) L$$

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By sending the empty link to 1, we can define an isomorphism

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Let $< l_1, l_2, ..., l_k >_{D(L)}$ be the complex number obtained by cabling $l_1, ..., l_k \in \mathrm{K}_r(A \times [0,1])$ along k ordered components of $D(L)$.

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On $\mathrm{K}_r(A \times [0,1])$, there is a commutative multiplication induced by the juxtaposition of annulus A, and as a $\mathbb{C}$-algebra

$$\mathrm{K}_r(A \times [0,1]) \cong \mathbb{C}[z]$$

where z is the core curve of A.

Skein approach of RT

Let $e_n(z)$ be the n-th Chebyshev polynomial (i.e., the Jones-Wenzl idempotent of Temperley-Lieb algebra) defined by the recursive relation

$$e_n(z) = ze_{n-1}(z) - e_{n-2}(z)$$

where $e_0(z) = 1$ and $e_1(z) = z$.

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The Kirby coloring $\omega_r \in \mathrm{K}_r(A \times [0,1])$ is then defined by

$$\omega_r = \sum_{n=0}^{r-2} (-1)^n [n+1] e_n$$

where $[n]$ is the quantum integer defined by

$$[n] = \frac{e^{\frac{2n\pi\sqrt{-1}}{r}} - e^{-\frac{2n\pi\sqrt{-1}}{r}}}{e^{\frac{2\pi\sqrt{-1}}{r}} - e^{-\frac{2\pi\sqrt{-1}}{r}}}$$

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U_+: the diagram of the unknot with framing 1; $\mu_r = \frac{\sin\frac{2\pi}{r}}{\sqrt{r}}$.

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Definition

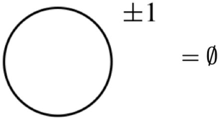
The r-th Reshetikhin-Turaev invariant of M_L is defined as

$$\mathrm{RT}_r(M_L) = \mu_r \langle \mu_r\omega_r, \ldots, \mu_r\omega_r \rangle_{D(L)} \langle \mu_r\omega_r \rangle_{U_+}^{-\sigma(L)}.$$

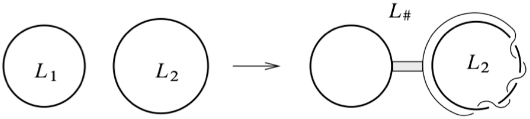
Reshetikhin-Turaev invariants

$\mathrm{RT}_r(M_L)$ is an invariant under Kirby moves (blow-up/down, handle slide).

$K1$:



$K2$:



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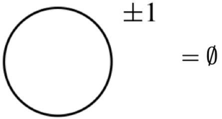
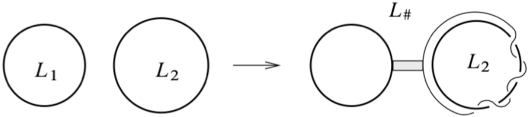


$K2:$



M_{L_1}, M_{L_2} are homeomorphic if and only if L_1 and L_2 are related by Kirby moves.

Lickorish-Wallace Theorem

Theorem (Lickorish-Wallace)

Any closed, orientable, connected 3-manifold can be obtained by integral Dehn surgery along a link in S^3.

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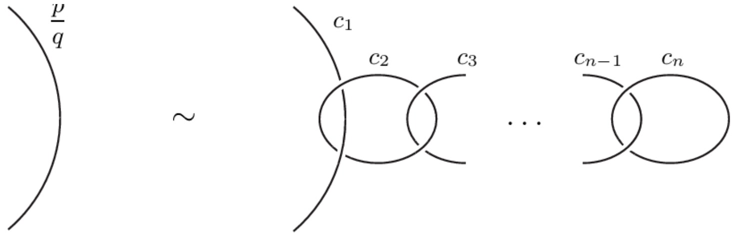
Theorem (Generate to cases with boundary)

Any compact, orientable, connected 3-manifold with boundary is obtained as follows: pick $L \cup C \subset S^3$ *where* L *is a link and* C *is a 1-complex; remove an open regular neighbourhood of* C *and perform an integral surgery along* L.

Kirby calculus

Convert Dehn surgery into integral surgery



Continued fractions

The rational Dehn cofficient p/q can be expressed as a continued fraction in the following manner:

$$p/q = a_k - \cfrac{1}{a_{k-1} - \cfrac{1}{\ddots - \frac{1}{a_1}}}$$

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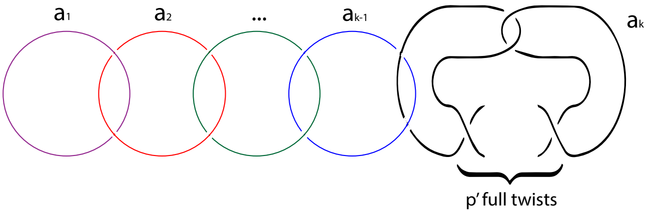
By continued fractions, we consider the closed manifold $\mathcal{K}_s(p,q)$ obtained by S^3 surgering along a framed link.



Calculation of RT Invariants

$$
\begin{aligned}
RT_r(\mathcal{K}_s(p,q)) =& \mu_r \left\langle \mu_r\omega_r, \ldots, \mu_r\omega_r \right\rangle_{D(L)} \left\langle \mu_r\omega_r \right\rangle_{U_+}^{-\sigma(L)} \\
=& \mu_r^{k+1} \left\langle \mu_r\omega_r \right\rangle_{U_+}^{-\sigma(L)} \left\langle \omega_r, \ldots, \omega_r \right\rangle_{D(L)}
\end{aligned}
$$

Let $t = e^{\frac{4\pi\sqrt{-1}}{r}}$, by direct computation

$$
\left\langle \mu_r\omega_r \right\rangle_{U_+} = e^{\left(-\frac{3}{r}-\frac{r+1}{4}\right)\pi\sqrt{-1}}
$$

$$
\begin{aligned}
\left\langle \omega_r, \ldots, \omega_r \right\rangle_{D(L)} =& \left\langle \sum_{n=0}^{r-2}(-1)^n[n+1]e_n, \ldots, \sum_{n=0}^{r-2}(-1)^n[n+1]e_n \right\rangle_{D(L)} \\
=& \sum_{m_1,\ldots,m_k=0}^{r-2} (-1)^{m_k+\sum_{i=1}^k a_i m_i}\, t^{\sum_{i=1}^k \frac{a_i m_i(m_i+2)}{4}} [m_1+1] \\
& \prod_{i=1}^{k-1}[(m_i+1)(m_{i+1}+1)] \langle e_{m_k} \rangle_{D_{\mathcal{K}_s}}
\end{aligned}
$$

Calculation of RT Invariants

$$\langle e_{m_k} \rangle_{D(\mathcal{K}_s)} = (-1)^{m_k}[m_k+1]J_{m_k+1}(\mathcal{K}_s)$$

$$= (-1)^{m_k}[m+1]\sum_{k'=0}^{m_k}\sum_{l=0}^{k'}(-1)^l t^{\frac{k'(k'+3)}{4}+s'l(l+1)}$$

$$\frac{\{2l+1\}\{k'\}!}{\{k'+l+1\}!\{k'-l\}!}.\prod_{i=1}^{k}\{m_k+1+i\}\{m_k+1-i\}$$

where $\{m\} = t^{m/2} - t^{-m/2}$, $\{m\}! = \prod_{k=1}^{m}\{k\}$, $k' =$

The second equal sign holds because we substitute **Masbum's cyclotomic expansion** of the colored Jones polynomial $J_{m_k+1}(\mathcal{K}_s)$ for the twist knot.

Cyclotomic expansion

Theorem (Kazuo Habiro)

The colored Jones polynomial can be rewritten into such expansion

$$\begin{aligned} J_N(K;q) =& 1 + H_1(q)\{N\}\{N+2\} + H_2(q)\{N-1\}\{N\}\{N+2\}\{N+3\} \\ & + \ldots + H_N(q)\{1\}\{2\}\ldots\{N\}\{N+2\}\ldots\{2N\}\{2N+1\} \end{aligned}$$

where $H_i(q) \in \mathbb{Z}[q^{\pm}], i = 1, 2, \ldots, N$

Theorem (Masbum)

The colored Jones polynomial of twist knot K_s is given by

$$J_N(K_s, q) = \sum_{n=0}^{\infty} f_{K_s,n} \frac{\{N-n\}\{N-n+1\}\ldots\{N+n\}}{\{N\}}$$

Poisson Summation Fomular

Definition (Poisson Summation Fomular)

Poisson summation Formula states that, for any function f in Schwarz space,

$$\sum_{(m_1,\ldots,m_k)\in\mathbb{Z}^k} f(m_1,\ldots,m_k) = \sum_{(n_1,\ldots,n_k)\in\mathbb{Z}^k} \hat{f}(n_1,\ldots,n_k)$$

where $\hat{f}(n_1,\ldots,n_k) = \int_{\mathbb{R}^k} f(x_1,\ldots,x_k)\, e^{-\sum_{j=1}^{k} 2\pi\sqrt{-1}n_j x_j}\, \mathrm{d}x_1\ldots\mathrm{d}x_k$

Calculation of RT

Finally we have:

$$\mathrm{RT}_r(\mathcal{K}_s(p,q)) = c_r \sum_{s=0}^{|q|-1} \sum_{(m,n,l)\in\mathbb{Z}^3} \check{f}_r(s,m,n,l) + o(e^{\frac{r}{4\pi}\cdot 3.5})$$

where

$$\check{f}_r(s,m,n,l) = \int_D g(s,x,y,z) e^{\frac{r}{4\pi i} V(x,y,z,s,m,n,l)} dxdydz(1+O(\frac{1}{r}))$$

and V is the potential function.

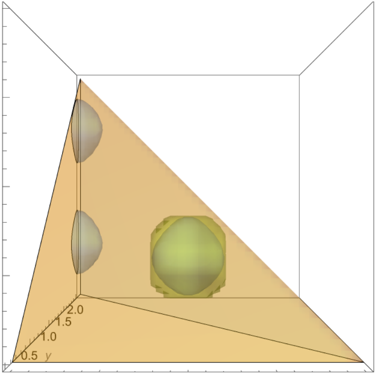
$$\begin{aligned}
V(x,y,z,s,m,n,l) &= \mathrm{Li}_2(e^{2i(y+x)}) + \mathrm{Li}_2(e^{2i(y-x)}) \\
&+ \mathrm{Li}_2(e^{2i(-y+z)}) + \mathrm{Li}_2(e^{2i(-y-z)}) - \mathrm{Li}_2(e^{-2iy}) \\
&- \frac{p+2q}{q}x^2 - 4y^2 - (4p'+2)z^2 + 2\pi k(s,m)x - 4\pi ny \\
&- 2\pi(2l+1)z - \frac{\pi^2}{2} + K(s)\pi^2
\end{aligned}$$

The "Big Cancellation"

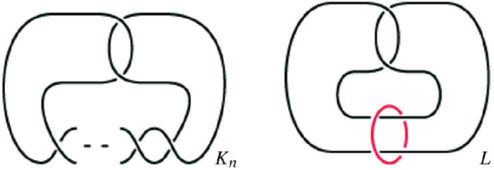
Qingtao Chen and **Shengmao Zhu** found the"Big Cancellation" of the Fourier terms . This cancellation also occurs in our case.

Proposition

$$\check{f}_r(s, m, n, l) = (-1)^{p'+l} \cdot \check{f}_r(s, m, n, -2p' - 2 - l)$$



Twist Knots and Whitehead Link

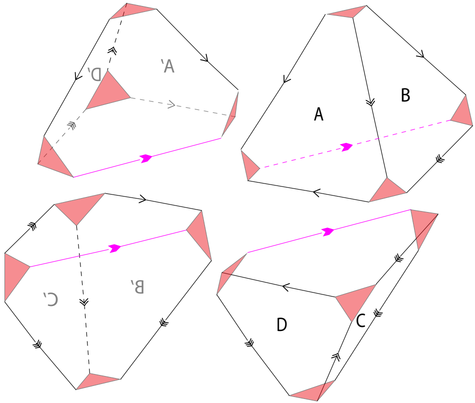


It is easy to see that twist knot $\mathcal{K}_s$ can be considered as a whitehead link getting $(1, -s)$-dehn surgery along one component.

s	0	1	-1	2	-2	...
$\mathcal{K}_s$	0_1	3_1	4_1	5_2	6_1	...
Coefficient	$(1, 0)$	$(1, -1)$	$(1, 1)$	$(1, -2)$	$(1, 2)$	...

$$K_s(p, q) \cong W((p, q), (1, -s))$$

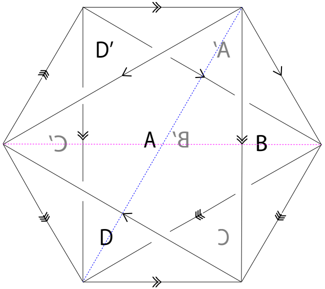
Thurston's triangulation



According to Thurston's book, Whitehead link's complement in S^3 can be canonically decomposited into 4 idea tetrahedra.

Thurston's triangulation

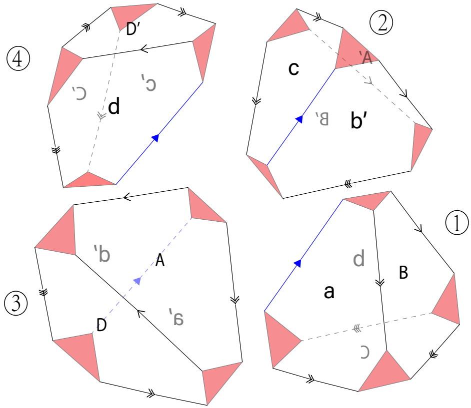
The Whitehead link complement can be viewed as an ideal octahedron with its faces glued together.



Thurston's triangulation: cut along the red line; Our triangulation: blue line.

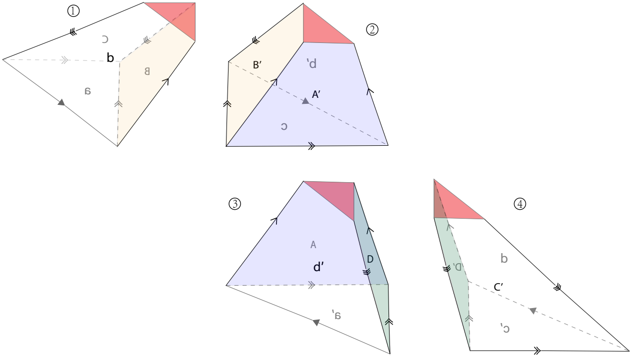
Our triangulation

A different approach to cut the octahedron into four tetrahedrons



Our triangulation

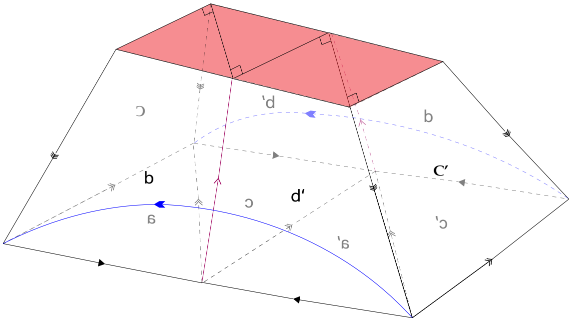
Glue the four tetrahedrons in a new way



We remove an edge and add an edge with long solid arrow.

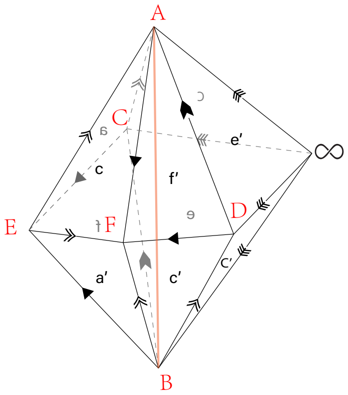
Our triangulation

We remove the red edge and add an blue edge. Now, this object becomes an pentagonal bipyramid.



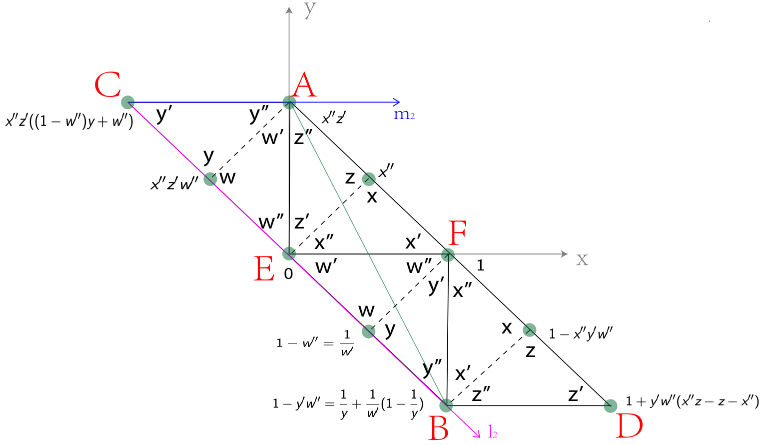
Our triangulation

The ideal pentagonal bipyramid.



Connect AB, then we obtained an idea triangulation containing 5 ideal tetrahedra.

Our triangulation



The geometry of the critical points of V

Consider Thurston's gluing equations of $W_{(-p-4q,q),((1,s-\frac{1}{2}))}$

$$\begin{cases} H(m_1) - (s - \dfrac{1}{2})H(l_1) = 2\pi i \\ (-p-4q)H(m_2) + qH(l_2) = 2\pi i \\ H(e_1) = H(e_2) = H(e_3) = H(e_4) = H(e_5) = 2\pi i \end{cases}$$

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and critical point equations of $V^+(x,y,z)$

$$\frac{\partial V^+(x,y,z)}{\partial x} = \frac{\partial V^+(x,y,z)}{\partial y} = \frac{\partial V^+(x,y,z)}{\partial z} = 0 \tag{2}$$

The geometry of the critical points of V

Proposition

The system of Thurston's gluing equations of $W_{((-p-4q,q),(1,s-\frac{1}{2}))}$ *is equivalent to the system of the critical point equations of* $V^{+}(x,y,z)$.

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Question

How to fix the discrepancy in the Dehn coefficients?
$W((-p-4q,q),(1,s-\frac{1}{2}))$ *instead of* $K_s(p,q) \cong W((p,q),(1,-s))$

Hodgson, Meyerhoff and Weeks' sister manifolds

Theorem (Hodgson, Meyerhoff, Weeks)

Let W be the Whitehead link complement, and p, q are relatively prime integers. $W_{(-p-4q,q),(1,s-\frac{1}{2})}$ and $W_{((p,q),(1,-s))}$ have equal volumes and Chern-Simons invariants.(With some requirements for Dehn coefficients)

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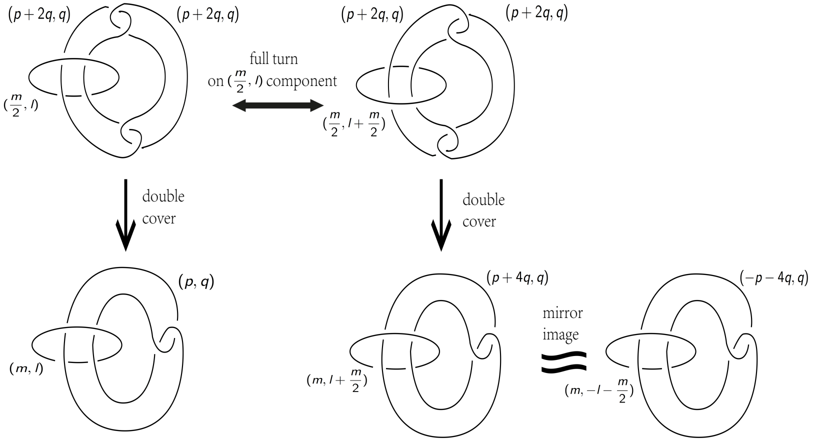
Note that $W_{(-p-4q,q),(1,s-\frac{1}{2})}$ has a cone-angle of 4π, which is not a normal hyperbolic manifold.

Hodgson, Meyerhoff and Weeks' sister manifolds



Sister potential function W

Definition

$W(x, y, z, s, m, n, l) = \mathrm{Li}_2(e^{2i(y+x)}) + \mathrm{Li}_2(e^{2i(y-x)}) + \mathrm{Li}_2(e^{2i(-y+z)}) + \mathrm{Li}_2(e^{2i(-y-z)}) - \mathrm{Li}_2(e^{-2iy}) + \frac{p+2q}{q}x^2 - 4y^2 + 4(p' - 1)z^2 + 2\pi k(s, m)x - 4\pi ny - 4\pi lz - \frac{\pi^2}{2} - K(s)\pi^2$

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Theorem

The system of critical point equations of $W(x,y,z,s^+,m^+,0,l^+)$ *is equivalent to the hyperbolic gluing equations of* $\mathcal{W}((p,q),(1,-s))$.

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Theorem

The system of critical point equations of $W(x, y, z, s^+, m^+, 0, l^+)$ is equivalent to the hyperbolic gluing equations of $\mathcal{W}((p, q), (1, -s))$.

Theorem

- (x, y_1, z) *is the critical point of* $V \Leftrightarrow (x, y_2, z)$ *is the critical point of* W;
- $V(x, y_1, z) + W(x, y_2, z) = 0$.

Geometry of the Critical Points and Critical Value

Theorem (Yoshida, 1985, Invent. Math.)

$$\sqrt{-1}\,\mathrm{Vol}(M) - \mathrm{CS}(M) = \Phi(\mathrm{H}(m_1), \ldots \mathrm{H}(m_k))$$
$$-\sum_{i=1}^{k} \frac{\mathrm{H}(m)\mathrm{H}(l)}{4} + \frac{\theta_i\sqrt{-1}\mathrm{H}(\gamma_i)}{4} \quad \left(\bmod \sqrt{-1}\pi^2\mathbb{Z}\right)$$

where Φ is the function defined on the deformation space of hyperbolic structures on $\mathrm{S}^3 \backslash L$ parametrized by $u_i = \mathrm{H}(m_i)$ characterized by

$$\begin{cases} \dfrac{\partial \Phi(u_i)}{\partial u} = \dfrac{\mathrm{H}(l)}{2}, \\ \Phi(0) = \sqrt{-1}\left(\mathrm{Vol}\left(\mathrm{S}^3\backslash L\right) + \sqrt{-1}\mathrm{CS}\left(\mathrm{S}^3\backslash L\right)\right) \quad \left(\bmod \sqrt{-1}\pi^2\mathbb{Z}\right). \end{cases}$$

Geometry of the Critical Points and Critical Value

Using the above Yoshida's fomula, we obtain the critical value $V^{+}(x_0, y_0, z_0)$,

Proposition

Let $\mathrm{Vol}(\mathcal{K}_s(p,q))$ *and* $\mathrm{CS}(\mathcal{K}_s(p,q))$ *are the hyperbolic volume and the Chern-Simons invariant of* $\mathcal{K}_s(p,q)$*, Then we have*

$$V^{+}(x_0, y_0, z_0) = \sqrt{-1}(\mathrm{Vol}(\mathcal{K}_s(p,q)) + \sqrt{-1}\mathrm{CS}(\mathcal{K}_s(p,q))) \pmod{\pi^2\mathbb{Z}}$$

A similar conclusion can be drawn for $V^{-}(x,y,z)$.

Saddle point method

Theorem (Ohtsuki)

Let D be a region in $\mathbb{C}^n$ and let f, g be holomorphic functions on D, S be an embedded real n-dimensional closed disk in D. If

(1) $(c_1, \ldots, c_n) \in S$ *is a critical point of* f *in* D*;*

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then we have

$$\int_S g(z_1, \ldots, z_n) e^{rf(z_1, \ldots, z_n)} dz_1 \ldots dz_n$$
$$= \left(\frac{2\pi}{r}\right)^{\frac{n}{2}} \frac{g(c_1, \ldots, c_n)}{\sqrt{-\det \mathrm{Hess}(f)(c_1, \ldots, c_n)}} e^{rf(c_1, \ldots, c_n)} \left(1 + O\left(\frac{1}{r}\right)\right)$$

Estimates of the Fourier Cofficients

In orange region $\Im(\mathrm{Hess}(V))$ is negative definite.

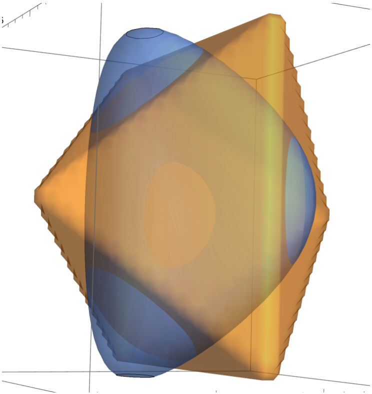

Figure: The region $\Im V(x, y, z) > 3.5$

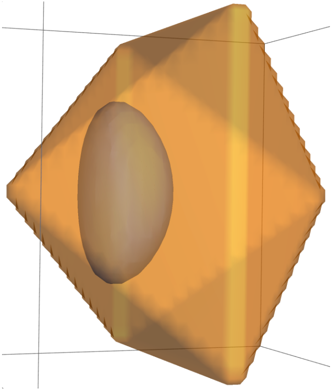

Figure: The region $\Im V(x, y + i\Im(y_0), z) > 3.5$

Estimates of the Fourier Cofficients

Estimating these Fourier coefficients using the saddle point method, we have

Theorem

If the volume $Vol(\mathcal{K}_s(p,q))$ corresponding to the coefficients (p,q,s) is greater than 3.55, then we have:

$$
\begin{aligned}
& h_{1,r}(s^+,-l^+,0,0)+h_{1,r}(s^-,-l^-,0,0) \\
+& h_{1,r}(s^+,-l^+,0,2)+h_{1,r}(s^-,-l^-,0,2) \\
=& 4(-1)^{l^+}(\frac{r}{2\pi})^3 a^+(x_0,y_0,z_0) e^{\frac{4\pi\sqrt{-1}}{r}(V^+(x_0,y_0,z_0))}(1+O(\frac{1}{r}))
\end{aligned}
$$

and

$$|h_{1,r}(s,l,0,n)| < O(e^{\frac{r}{4\pi}(Vol(\mathcal{K}_s(p.q))-\varepsilon)})$$

and

$$|h_{1,r}(s,l,m,n)| < O(e^{\frac{r}{4\pi}(3)}) \qquad m \neq 0$$

Asymptotics of the RT&TV Invariants

Theorem (Ge-Meng-W.-Yang, 2024)

Let $M = \mathcal{K}_s(p,q)$ be the manifold obtained by S^3 doing p/q Dehn-Surgery along the twist knot $\mathcal{K}_s$. Then there exists a N ($N = 10$ is enough.), so that if $|p|, |q|, |s| > N$,

$$RT_r(\mathcal{K}_s(p,q)) = C_r I_{\mathcal{K}_s(p,q)} e^{\frac{r}{4\pi} Vol_c(\mathcal{K}_s(p,q))}(1 + O(\frac{1}{r}))$$

$$TV_r(\mathcal{K}_s(p,q)) = 2|I_{\mathcal{K}_s(p,q)}|^2 e^{\frac{r}{2\pi} Vol(\mathcal{K}_s(p,q))}(1 + O(\frac{1}{r}))$$

where $|C_r| = 1$ and $I_{\mathcal{K}_s(p,q)}$ is an invariant

$$Vol_c(\mathcal{K}_s(p,q)) = Vol(\mathcal{K}_s(p,q)) + \sqrt{-1}CS(\mathcal{K}_s(p,q))(mod\sqrt{-1}\pi^2\mathbb{Z})$$

Chen-Yang Volume Conjecture for the twist knot

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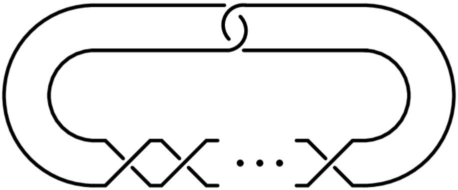
$$\lim_{r\to\infty, r \text{ odd}} \frac{4\pi}{r} \log \mathrm{RT_r}(\mathcal{K}_s(p,q))$$
$$= \mathrm{Vol}(\mathcal{K}_s(p,q)) + \sqrt{-1}\,\mathrm{CS}(\mathcal{K}_s(p,q)) \ (mod \sqrt{-1}\pi^2\mathbb{Z}),$$



Bound for N

Question

Why we need $|p|, |q|, |s| > N$

Bound for N

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Why we need $|p|, |q|, |s| > N$

Theorem (Materlli)

Dehn Fillings on Whitehead link yeilds a closed hyperbolic manifold $M = W_{(\frac{p_1}{q_1}, \frac{p_2}{q_2})}$ *unless one of the following holds*

- $\frac{p_1}{q_1}$ *or* $\frac{p_2}{q_2} \in \{0, 1, 2, 3, 4, \infty\}$.
- $(\frac{p_1}{q_1}, \frac{p_2}{q_2}) \in \{(-4, 1), (-3, 1), (-2, -2), (-2, -1), (\frac{3}{2}, 5), (\frac{4}{3}, 5), (\frac{5}{2}, \frac{7}{2})\}$ *up to permutation.*

Bound for N

Question

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We need to make sure there is a solution to the hyperbolic gluing equation with Dehn cofficients $(p, q), (1, -s)$ and $(-4p - q, q), (1, s - 1/2)$.

Bound for N

In our estimation we need to ensure that the $Vol(\mathcal{K}_s(p,q)) > 3.55$. And $N = 10$ is enough.

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An interesting lemma

Lemma

Suppose
$$\vec{\mathbf{z}} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n, \vec{\mathbf{r}} = (r_1, \cdots, r_m) \in \mathbb{R}^m, L = \begin{pmatrix} \vec{l_1} \\ \vdots \\ \vec{l_m} \end{pmatrix} \in \mathbb{R}^{m \times n},$$
p is a n-variables real polynomial with degree ≤ 2 *and*

$$f(\vec{\mathbf{z}}) = \vec{\mathbf{r}} \begin{pmatrix} \mathrm{Li}_2(e^{i\vec{l_1}\vec{\mathbf{z}}}) \\ \vdots \\ \mathrm{Li}_2(e^{i\vec{l_m}\vec{\mathbf{z}}}) \end{pmatrix} + p(\vec{\mathbf{z}})$$

An interesting lemma

then we have

$$\Im f(\vec{\mathbf{z}}) = \vec{\mathbf{r}} \begin{pmatrix} \mathrm{D}_2(e^{i\vec{l_1}\vec{\mathbf{z}}}) \\ \vdots \\ \mathrm{D}_2(e^{i\vec{l_m}\vec{\mathbf{z}}}) \end{pmatrix} + (\Re \nabla f(\vec{\mathbf{z}}))\Im \vec{\mathbf{z}}$$

particularly, if $\vec{\mathbf{z_0}}$ is a critical point of f ($\nabla f(\vec{\mathbf{z_0}}) = \vec{0}$), then we have

$$\Im f(\vec{\mathbf{z_0}}) = \vec{\mathbf{r}} \begin{pmatrix} \mathrm{D}_2(e^{i\vec{l_1}\vec{\mathbf{z_0}}}) \\ \vdots \\ \mathrm{D}_2(e^{i\vec{l_m}\vec{\mathbf{z_0}}}) \end{pmatrix}$$

Related research

$SU(n)$-invariants

Theorem (Ge-W.)

The $SU(n)$-invariants of torus knot $T(p,q)$ has a cyclotomic expansion with p and N small.

Related research

$SU(n)$-invariants

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Relative quantum invariants
in preparation

Thank You

Hyperbolic Gluing Equations

We computed the **potential function** of $\mathcal{K}_s$ as

$$V(s, x, y, z, l, m, n) = \lim_{r\to\infty} V_r(s, x, y, z) + F(l, m, n)$$

There are some dilogarithm function Li_2 terms in $V(s, x, y, z, l, m, n)$.
We define

$$\begin{aligned}
&x'' z' ((1 - w'')y + w'') \\
&x'' z' \\
&x'' z' w'' \\
&x'' \\
&0 \\
&1 \\
&1 - w''^2
\end{aligned}$$

where $s^{\pm}$, $l^{\pm}$ come from the algebra of continued fractions.